

The separating gonality of a separating real curve

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Abstract

Let X be a smooth real curve of genus g such that the real locus has s connected components. We say X is separating if the complement of the real locus is disconnected. In case there exists a morphism f from X to \mathbb{P}^1 such that the inverse image of the real locus of \mathbb{P}^1 is equal to the real locus of X then X is separating and such morphism is called separating. The separating gonality of a separating real curve X is the minimal degree of a separating morphism from X to \mathbb{P}^1 . It is proved by Gabard that this separating gonality is between s and $(g + s + 1)/2$. In this paper we prove that all values between s and $(g + s + 1)/2$ do occur.

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1 Introduction

Let X be a smooth real curve of genus g . We assume X is complete and geometrically irreducible, hence its set $X(\mathbb{C})$ of complex points is in a natural way a compact Riemann surface of genus g . Let $X(\mathbb{R})$ be the set of real points, it is well known that $X(\mathbb{C}) \setminus X(\mathbb{R})$ is either connected or it has two connected components. In case it has two connected components then we say X is a *separating real curve*.

It is well known that each component of $X(\mathbb{R})$ is a smooth analytic real manifold of dimension 1 homeomorphic to a circle. Let $s = s(X)$ be the number of connected components of $X(\mathbb{R})$. In case X is separating one knows $1 \leq s \leq g + 1$ and $s \equiv g + 1 \pmod{2}$. For each integer s satisfying those conditions there exists a real separating curve X of genus g with $s(X) = s$.

Let $f : X \rightarrow \mathbb{P}^1$ be a morphism such that $f^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R})$. Hence each real fiber of f only consists of real points, such morphism is called *totally real* or *separating*. We use the terminology of separating: since $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is not connected the existence of such morphism f implies X is separating. Conversely, it is proved in [1] that for each separating real curve X of genus g there exists a separating morphism of degree at most $g + 1$. In [6] one proves a stronger

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result: each separating real curve X of genus g with $s = s(X)$ has a separating morphism of degree at most $(g + s + 1)/2$. At the end of the introduction of that paper, the author mentions that sharpness of that upper bound is an open problem. Related to this problem and copying the terminology of gonality of a complex curve, we introduce the following definition.

Definition 1. *Let X be a separating real curve of genus g . The separating gonality of X is the minimal degree k such that there exists a separating morphism $f : X \rightarrow \mathbb{P}^1$ of degree k . This separating gonality is denoted by $\text{sepgon}(X)$.*

The result in [6] can be written as $\text{sepgon}(X) \leq (g + s + 1)/2$ with $s = s(X)$. Moreover, there is the trivial lower bound $\text{sepgon}(X) \geq s$. In this paper we prove the following theorem, giving amongst others an affirmative answer to the question on the sharpness on the upper bound for sepgon .

Theorem 2. *Let g , s and k be integers satisfying $g \geq 2$, $1 \leq s \leq g + 1$ with $s \equiv g + 1 \pmod{2}$ and $\max\{2, s\} \leq k \leq (g + s + 1)/2$. Then there exists a separating real curve X of genus g with $s(X) = s$ and $\text{sepgon}(X) = k$.*

In case $s = g + 1$ then the lower and upper bound on sepgon agree. Such curves are called M-curves and separating morphisms of degree $g + 1$ on M-curves are studied in [8]. As a matter of fact, if C_1, \dots, C_{g+1} are the connected components of the real locus of an M-curve X and $P_i \in C_i$ for $1 \leq i \leq g + 1$ then $\{P_1, \dots, P_{g+1}\}$ is the fiber of a separating morphism $f : X \rightarrow \mathbb{P}^1$, uniquely determined by those points up to an automorphism of \mathbb{P}^1 . However the other extreme case $s = 1$ is the base case for the proof of the theorem. In this case the upper bound on sepgon is equal to the Meis bound for the gonality of compact Riemann surfaces ([9]) and this is nowadays contained in Brill-Noether Theory (see e.g. [3, Chapter V]). Using results concerning pencils on Riemann surfaces we obtain a proof of the theorem in case $s = 1$ (this argument resembles those used in [4, Section 3]). Then we take a suited separating real curve X' with $s(X') = 1$ and on $X'_\mathbb{C}$ we identify closed points associated to some chosen non-real points on X' . This gives rise to a singular real curve X_0 on which each identification gives rise to an isolated real node. We consider a suited smoothing X_t of X_0 defined over \mathbb{R} such that each isolated real node becomes a component of $X_t(\mathbb{R})$ and such that X_t has a suited separating morphism to \mathbb{P}^1 . Such smoothing X_t is a separating real curve with $s(X_t) = s$ and we prove that if X_t is close enough to X_0 then it has the desired separating gonality.

2 Notations and preliminaries

Let \mathbb{R} be the field of real numbers, let X be an indeterminate over \mathbb{R} and let \mathbb{C} be the field $\mathbb{R}[X]/(X^2 + 1)$, called the field of complex numbers. The complex number defined by X is denoted by i ; in this way we have the field extension $\mathbb{R} \subset \mathbb{C}$. For a complex number $z = x + iy$ ($x, y \in \mathbb{R}$), its complex conjugate is denoted by $\bar{z} = x - iy$.

A real scheme is a reduced, separated scheme of finite type over \mathbb{R} . In case V is a real scheme then $V_{\mathbb{C}}$ is the scheme obtained from V by the base change defined by $\mathbb{R} \subset \mathbb{C}$. We say V is smooth, irreducible, ... in case $V_{\mathbb{C}}$ is smooth, irreducible, We write $V(\mathbb{C})$ to denote the set of closed points of $V_{\mathbb{C}}$. On $V_{\mathbb{C}}$, hence also on $V(\mathbb{C})$, we have the \mathbb{R} -morphism induced by complex conjugation on \mathbb{C} . It is called complex conjugation on $V_{\mathbb{C}}$ and on $V(\mathbb{C})$. In case $P \in V(\mathbb{C})$ then \overline{P} is the point obtained by complex conjugation. There are two types of closed points on V : those corresponding to $P \in V(\mathbb{C})$ with $P = \overline{P}$ and those corresponding to $\{P, \overline{P}\} \subset V(\mathbb{C})$ with $P \neq \overline{P}$. In case $P = \overline{P}$ then P is called a *real point* on V . The set of real points on V is denoted by $V(\mathbb{R})$, it is a subset of $V(\mathbb{C})$. In case $P \neq \overline{P}$ then the associated point on V is called a non-real point on V and it is denoted by $P + \overline{P}$. In case V and W are two real schemes then a morphism $f : V \rightarrow W$ is defined over \mathbb{R} . We write $f_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ to denote the morphism defined by the field extension $\mathbb{R} \subset \mathbb{C}$. It induces maps $f(\mathbb{C}) : V(\mathbb{C}) \rightarrow W(\mathbb{C})$ and $f(\mathbb{R}) : V(\mathbb{R}) \rightarrow W(\mathbb{R})$. We write \mathbb{P}^N to denote the scheme $\text{Proj}(\mathbb{R}[X_0, \dots, X_N])$ (see [7]). It is called the real projective space of dimension N . In accordance to the general notations then $\mathbb{P}_{\mathbb{C}}^N$ is the complex projective space of dimension N and we have an inclusion $\mathbb{P}^N(\mathbb{R}) \subset \mathbb{P}^N(\mathbb{C})$.

For some basic terminology used now we refer to [7]. A real scheme X is called a real curve if $X_{\mathbb{C}}$ is a connected, reduced curve (hence it has dimension 1 but it can be singular); X is called complete, irreducible, smooth, stable, ... in case $X_{\mathbb{C}}$ is complete, irreducible, smooth, stable Now assume X is a real complete curve. An \mathcal{O}_X -Module L that is locally isomorphic to \mathcal{O}_X is called an invertible sheaf on X . Using the field extension $\mathbb{R} \subset \mathbb{C}$ we obtain an invertible sheaf $L_{\mathbb{C}}$ on $X_{\mathbb{C}}$. The space of global sections $\Gamma(X, L)$ (resp. $\Gamma(X_{\mathbb{C}}, L_{\mathbb{C}})$) is a finite dimensional \mathbb{R} -vectorspace (resp. \mathbb{C} -vectorspace). Complex conjugation on X induces a complex conjugation on $\Gamma(X_{\mathbb{C}}, L_{\mathbb{C}})$ having invariant space $\Gamma(X, L)$. In this way we consider $\Gamma(X_{\mathbb{C}}, L_{\mathbb{C}})$ as the complexification of $\Gamma(X, L)$ and we denote $h^0(L)$ for both $\dim_{\mathbb{R}}(\Gamma(X, L))$ and $\dim_{\mathbb{C}}(\Gamma(X_{\mathbb{C}}, L_{\mathbb{C}}))$. A Cartier divisor D on X defines an invertible sheaf $\mathcal{O}_X(D)$ and it is well-known that $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))$ (resp. $\mathbb{P}(\Gamma(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)))$) parameterizes the space of effective divisors on X (resp. $X_{\mathbb{C}}$) linearly equivalent to D . This is called the complete linear system defined by D , it is denoted by $|D| = \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))$ and $|D|_{\mathbb{C}} = \mathbb{P}(\Gamma(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)))$. A linear subspace V of $\Gamma(X, \mathcal{O}_X(D))$ defines a linear subsystem $\mathbb{P}(V)$ of $|D|$. In case $\deg(D) = d$ and $r+1 = \dim_{\mathbb{R}}(V)$ then we say $\mathbb{P}(V)$ is a g_d^r on X . On $X_{\mathbb{C}}$ we have $\mathbb{P}(V)_{\mathbb{C}} = \mathbb{P}(V_{\mathbb{C}})$ (here $V_{\mathbb{C}} \subset \Gamma(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D))$) is the complexification of V and $g_{d,\mathbb{C}}^r$. Of course using subvectorspaces of the global space of an invertible sheaf L on $X_{\mathbb{C}}$ we obtain linear systems on $X_{\mathbb{C}}$ (but not necessarily defined by linear systems on X).

The moduli functor of stable curves of genus g is not representable, hence there is no fine module scheme with a universal family. Instead we make use of so-called suited families of stable curves.

Definition 3. *Let X be a complete stable real curve of genus g . A suited family of stable curves of for X is a projective morphism $\pi : \mathcal{C} \rightarrow S$ defined over \mathbb{R} such that*

1. $S_{\mathbb{C}}$ is smooth, irreducible and quasi-projective.
2. Each fiber of $\pi_{\mathbb{C}}$ is a stable curve of genus g .
3. For each $s \in S(\mathbb{C})$ the Kodaira-Spencer map $T_s(S_{\mathbb{C}}) \rightarrow \text{Ext}^1(\Omega_{\pi_{\mathbb{C}}^{-1}(s)}, \mathcal{O}_{\pi_{\mathbb{C}}^{-1}(s)})$ is surjective (here $\Omega_{\pi_{\mathbb{C}}^{-1}(s)}$ is the sheaf of Kähler differentials).
4. There exists $s_0 \in S(\mathbb{R})$ such that X is isomorphic to $\pi^{-1}(s_0)$.

From the third condition it follows that for a suited family $\pi : \mathcal{C} \rightarrow S$ of stable curves of genus g one has $\dim(S) = 3g - 3$ in case $g \geq 2$.

Lemma 4. *Let X be a complete stable curve of genus $g \geq 2$. There exists a suited family of stable curves for X .*

Proof. We are going to use some facts mentioned in [5, Section 1]). The moduli functor of tricanonically embedded stable curves of genus g is represented by a quasi-projective scheme \mathcal{H} defined over \mathbb{R} and there exists $x \in \mathcal{H}(\mathbb{R})$ corresponding to a tricanonical embedding of X . This scheme is smooth and moreover the natural transformation to the moduli functor of stable curves obtained by forgetting the embedding of the fibers is formally smooth. Hence the induced Kodaira-Spencer map $T_x(\mathcal{H}) \rightarrow \text{Ext}^1(\Omega_X, \mathcal{O}_X)$ (the tangent space of the moduli functor of stable curves at X) is surjective. Taking an embedding of \mathcal{H} in some projective space and intersecting with suited hyperplanes through x one obtains a suited family of stable curves for X . \square

As already mentioned in the introduction, we are going to use singular real curves obtained by making real isolated singular points out of some fixed non-real points. First we prove a lemma on the canonical embedding of those singular curves that we need in the proof. A lot of geometry of a canonically embedded curve is determined by the Riemann-Roch Theorem, so first we recall this theorem.

Let X be a real nodal curve of arithmetic genus g . It has a dualizing sheaf ω_X (defined over \mathbb{R}). This is an invertible sheaf on X and it satisfies the Riemann-Roch Theorem: for an invertible sheaf L on X one has

$$h^0(L) = \deg(L) - g + 1 + h^0(\omega_X \otimes L^{-1})$$

(see e.g. [10, Chapter 4]). In particular ω_X is the unique invertible sheaf of degree $2g - 2$ whose space of global sections has dimension g . Now let $1 \leq m \leq g - 2$ be an integer and let X' be a smooth real curve of genus $g' = g - m$ and for $1 \leq i \leq m$ let $P_i + \overline{P_i}$ be different non-real points on X' such that $\dim |P_i + \overline{P_i}| = 0$ (such non-real points exist because X' has at most one g_2^1 and this cannot contain all non-real points). Let X be the singular nodal real curve obtained from X' by identifying P_i with $\overline{P_i}$ for $1 \leq i \leq m$. This identification gives rise to an isolated real singular point S_i on X . The arithmetic genus of X is g .

Lemma 5. *ω_X is very ample.*

Proof. For $1 \leq j \leq m$ let X_j be the curve obtained from X' by identifying P_i with \overline{P}_i for $1 \leq i \leq j$ and again write S_i for the resulting nodes on X_j . One has $X_m = X$ and we write X_0 instead of X' . So for each $1 \leq j \leq m$ the curve X_j is obtained from the curve X_{j-1} by identifying P_j and \overline{P}_j considered as points on X_{j-1} . First we prove ω_{X_1} is very ample on X_1 and then, taking $2 \leq j \leq m$ and assuming $\omega_{X_{j-1}}$ is very ample on X_{j-1} we prove ω_{X_j} is very ample on X_j .

From the Riemann-Roch Theorem on X_0 it follows $h^0(\omega_{X_0}(P_1 + \overline{P}_1)) = g' + 1$ and for all $Q \in X_0(\mathbb{C})$ one has $h^0(\omega_{X_0, \mathbb{C}}(P_1 + \overline{P}_1 - Q)) = g'$. This proves the complete linear system $|\omega_{X_0}(P_1 + \overline{P}_1)|_{\mathbb{C}}$ on $X_{0, \mathbb{C}}$ is base point free, hence it defines a morphism $i_1 : X_0 \rightarrow \mathbb{P}^{g'}$ having image $i_1(X_0)$. Since $h^0(\omega_{X_0}) = g' = h^0(\omega_{X_0}(P_1 + \overline{P}_1)) - 1$ it follows $i_1(\mathbb{C})(P_1) = i_1(\mathbb{C})(\overline{P}_1) = S'_1$ is a real singular point on $i_1(X_0)$. Since $g' \neq 0$ for $Q \in X_0(\mathbb{C})$ one has $h^0(\mathcal{O}_{X_0, \mathbb{C}}(Q)) = 1$, hence $h^0(\omega_{X_0, \mathbb{C}}(-Q)) = g - 1 = h^0(\omega_{X_0}(P_1 + \overline{P}_1)) - 2$. This proves $i_1(\mathbb{C})^{-1}(S'_1) = \{P_1, \overline{P}_1\}$ and $i_1(\mathbb{C})$ is a local immersion at both P_1 and \overline{P}_1 . Projection from $\mathbb{P}^{g'}$ to $\mathbb{P}^{g'-1}$ with center S'_1 induces the canonical map $i_0 : X_0 \rightarrow \mathbb{P}^{g'-1}$. Since $P_1 + \overline{P}_1 \notin g_2^1$ (if X_0 would have a g_2^1) it follows $i_0(\mathbb{C})(P_1) \neq i_0(\mathbb{C})(\overline{P}_1)$, hence S'_1 is an ordinary node on $i_1(X_0)$. In case $i_1(X_0)_{\mathbb{C}}$ would have another singular point S' then there exists an effective divisor F of degree 2 on $X_0(\mathbb{C})$ having disjoint support with $P_1 + \overline{P}_1$ such that $h^0(\omega_{X_0, \mathbb{C}}(P_1 + \overline{P}_1 - F)) = g'$. Because of the Riemann-Roch Theorem it would imply $\omega_{X_0, \mathbb{C}}(P_1 + \overline{P}_1 - F) \cong \omega_{X_0, \mathbb{C}}$ hence F and $P_1 + \overline{P}_1$ are linearly equivalent. This again contradicts $P_1 + \overline{P}_1 \notin g_2^1$. So we obtain $i_1(X_0) = X_1$ and $X_1 \subset \mathbb{P}^{g'}$ is embedded by an invertible sheaf on X_1 of degree $2g' = 2p_a(X_1) - 2$ since $p_a(X_1) = g' + 1$. It follows $X_1 \subset \mathbb{P}^{g'}$ is canonical embedded, hence ω_{X_1} is very ample.

Now, let $2 \leq j \leq m$ and assume $\omega_{X_{j-1}}$ is very ample. Let $i_{j-1} : X_{j-1} \rightarrow \mathbb{P}^{g'+j-2}$ be a canonical embedding. Consider $\omega_{X_{j-1}}(P_j + \overline{P}_j)$ on X_{j-1} . From the Riemann-Roch Theorem on X_{j-1} it follows the complete linear system $|\omega_{X_{j-1}}(P_j + \overline{P}_j)|$ is base point free, hence it defines a morphism $i_j : X_{j-1} \rightarrow \mathbb{P}^{g'+j-1}$. One finds again $i_j(P_j) = i_j(\overline{P}_j) = S'_j$ is a real singular point on $i_j(X_{j-1})$ and projection of $\mathbb{P}^{g'+j-1}$ to $\mathbb{P}^{g'+j-2}$ with center S'_j induces the canonical embedding i_{j-1} of X_{j-1} . This implies S'_j is an ordinary node and $i_j(\mathbb{C})$ defines an isomorphism between $X_{j-1} \setminus \{P_j, \overline{P}_j\}$ and $i_j(X_{j-1}) \setminus \{S'_j\}$. This proves $X_j \cong i_j(X_{j-1})$ and as before we conclude $X_j \subset \mathbb{P}^{g'+j-1}$ is canonically embedded. \square

Remark. This lemma and its proof corresponds to the well-known description of the dualizing sheaf on a nodal curve as it is described in e.g. [10, Chapter IV.9]. According to that description a section of the dualizing sheaf of X' corresponds to a rational differential form on the normalization X having poles of order at most one on the inverse images of the nodes such that the sum of their residues at those 2 points is 0 and being regular outside the nodes. So the sections belong to $\Gamma(X, \omega_X(\sum_{i=1}^m (P_i + \overline{P}_i)))$ and if such section is 0 at some P_i (or \overline{P}_i) (meaning it is regular as differential form at P_i (or \overline{P}_i)) then because of the restriction on the residus, it need to be regular as differential forms on both P_i and \overline{P}_i , hence the section vanishes at both P_i and \overline{P}_i .

3 Proof of the theorem

First we are going to prove the theorem in case $s = 1$. Since X is separated it follows $g(X)$ is even, say $2h$. In this case $\text{sepgon}(X) = s = 1$ is impossible hence $2 \leq \text{sepgon}(X) \leq (g + 2)/2 = h + 1$. This upper bound is equal to the upper bound on the gonality of complex curves of genus $2h$. For $2 \leq k \leq h + 1$ it is proved in [4, Theorem 2] that there exists a real curve X of topological type $(2h, 1, 0)$ having a morphism $f : X \rightarrow \mathbb{P}^1$ of degree k such that $f(\mathbb{R}) : X(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$ is an unramified covering of degree k . For $k \leq h$ one has $2k - 2h - 2 < 0$ (hence the Brill-Noether Number $\rho_k^1(2h)$ (see [3, Chapter IV]) is negative) and therefore it follows from [4, Theorem 9] that a general real curve X of topological type $(2h, 1, 0)$ does not have a separating morphism of degree k . This proves the upperbound on sepgon obtained in [6] is sharp in case $s = 1$. In [6, Remark 11] it is mentioned that there exist real curves of topological type $(2h, 1, 0)$ having a separating morphism of degree k and not having a separating morphism of degree less than $k - 1$. This finishes in case $s = 1$ the proof of our theorem. Since this case is crucial for the proof of our theorem for all other cases we give some more details concerning those claims in [4, Remark 11].

Lemma 6. *Let h, k be integers with $h \geq 1$ and $2 \leq k \leq h$. There exists a real curve X of topological type $(2h, 1, 0)$ such that X has a separating morphism $f : X \rightarrow \mathbb{P}^1$ of type k and $X_{\mathbb{C}}$ has no linear system g_l^1 with $l < k$.*

Proof. From [4, Theorem 2], it follows there exists a real curve X' of topological type $(2h, 1, 0)$ having a separating morphism $f' : X' \rightarrow \mathbb{P}^1$. Let $\pi : \mathcal{C} \rightarrow S$ be a suited family of curves of genus g for X' . Choose $s' \in S(\mathbb{R})$ with $\pi^{-1}(s') = X'$. As in [4] we consider the S -scheme $\pi_k : H_k(\pi) \rightarrow S$ parameterizing morphisms of degree k from fibers of π to \mathbb{P}^1 . It is a smooth quasi-projective variety of dimension $2g + 2k - 2$ defined over \mathbb{R} . In particular $H_k(\pi)(\mathbb{R})$ parameterizes real morphisms of degree k from real fibers of π to \mathbb{P}^1 and this is a real analytic manifold (not connected) of real dimension $2g + 2k - 2$. In $\pi_k^{-1}(s')$ there is a point $[f']$ on $H_k(\pi)(\mathbb{R})$ corresponding to f' . It belongs to a connected component H of $H_k(\pi)(\mathbb{R})$ and since the topological degree (see [4]) of real morphisms is a discrete continuous invariant, it is constant on H . In our situation this means each point of H represents a separating morphism of degree k on a real curve of topological type $(2h, 1, 0)$ (also this topological type cannot change in a connected family of real curves). The dimension of $H_k(\pi)$ is in accordance with the Hurwitz formula for the number of ramification points of a covering $f_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow \mathbb{P}^1$ of degree k with $g(X_{\mathbb{C}}) = g$ (see e.g. [7, Chapter IV, Corollary 2.4]). Such covering is called simple in case all ramification points have index two and no two of them map to the same point of $\mathbb{P}_{\mathbb{C}}^1$. A moduli count using ramification shows that the subspace $H_k^{ns}(\pi)$ parameterizing non-simple coverings has dimension less than $2g + 2k - 2$. Since it is invariant under complex conjugation, it is defined over \mathbb{R} and therefore $\dim H_k^{ns}(\pi)(\mathbb{R}) < 2g + 2k - 2$ hence $H \not\subseteq H_k^{ns}(\pi)(\mathbb{R})$. Let $l < k$ and consider $H_{k,l} = H_k(\pi) \times_S H_l(\pi)$ and let $H'_{k,l}(\pi)$ be the image of $H_{k,l}$ on $H_k(\pi)$. This is a constructible subset of $H_k(\pi)$ and $H'_{k,l}(\pi)$ is invariant under complex conjugation, so it is defined over \mathbb{R} . Assume $([g], [h]) \in (H_{k,l})_{\mathbb{C}}$ and g

is simple. This defines a morphism $(g, h) : X \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ (here X is the complex fiber of π associated to $[g]$). Assume the image is not birational equivalent to X , then it defines non-trivial morphisms $g' : X \rightarrow X'$ and $g'' : X' \rightarrow \mathbb{P}_{\mathbb{C}}^1$ with $g = g'' \circ g'$. Since g is simple, g'' can not have ramification, a contradiction to the Hurwitz formula. So the image of (g, h) is birationally equivalent to X . Then [6, Proposition 2.4] implies $\dim_{([g],[h])}(H_{k,l}) = g + 2k + 2l - 7$. Since $l \leq h = g/2$ it follows $\dim_{([g],[h])}(H_{k,l}) < 2g + 2h - 2$. This proves $\dim(H'_{k,l}(\pi)) < 2g + 2k - 2$ and so $H \not\subseteq H'_{k,l}(\pi)(\mathbb{R})$. Take $[f] \in H$ with $[f] \notin H'_{k,l}(\pi)(\mathbb{R})$ for all $l < k$ and $s = \pi_k([f]) \in S(\mathbb{R})$ then $\pi^{-1}(s)$ is a real curve of topological type $(2h, 1, 0)$ and $f : \pi^{-1}(s) \rightarrow \mathbb{P}^1$ is a separating morphism of degree k and $\pi^{-1}(s)_{\mathbb{C}}$ has no morphism $g : \pi^{-1}(s)_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ of degree $l < k$. \square

Now assume $s > 1$. For $s \leq g + 1$ and $s \equiv g + 1 \pmod{2}$ it is proved in [4, Theorem 2] that there exists a real curve X of topological type $(g, s, 0)$ having a separating morphism $f : X \rightarrow \mathbb{P}^1$ of degree s . Of course, for such curve we have $\text{sepgon}(X) = s$, hence we can restrict to the case $s + 1 \leq k \leq (g + s + 1)/2$ (in particular also $s \leq g - 1$). Let $2h = g - s + 1$ and $k' = k - s + 1$, hence $2h \geq 2$ and $2 \leq k' \leq (g - s + 1)/2 + 1 = h + 1$. Take a real curve X' of topological type $(2h, 1, 0)$ such that $\text{sepgon}(X') = k'$ and $X'_{\mathbb{C}}$ has no morphism to $\mathbb{P}_{\mathbb{C}}^1$ of degree $l < k'$. Let $f' : X' \rightarrow \mathbb{P}^1$ be a separating morphism of degree k' . Choose general points P_2, \dots, P_s on $X'(\mathbb{C})$. Take $s - 1$ copies of \mathbb{P}^1 denoted by $(\mathbb{P}^1)_2, \dots, (\mathbb{P}^1)_s$ and let Γ_0 be the real curve obtained from $X' \cup (\mathbb{P}^1)_2 \cup \dots \cup (\mathbb{P}^1)_s$ by identifying P_i to $f'(P_i)$ and $\overline{P_i}$ to $f'(\overline{P_i}) = \overline{f'(P_i)}$ for $2 \leq i \leq s$. This singular real curve Γ_0 has arithmetic genus g and the morphism f' together with the identities between $(\mathbb{P}^1)_i$ and \mathbb{P}^1 for $2 \leq i \leq s$ give rise to a morphism $f_0 : \Gamma_0 \rightarrow \mathbb{P}^1$ defined over \mathbb{R} of degree k . Moreover for $P \in \mathbb{P}^1(\mathbb{R})$ the fiber $f_0^{-1}(P)$ only has real points, hence f_0 is a separating morphism for this singular real curve. The associated stable curve X_0 is the curve obtained from X' by identifying P_i and $\overline{P_i}$ for $2 \leq i \leq s$, hence it has a real isolated node S_i for $2 \leq i \leq s$ as its only singularities.

Locally at the image of P_i on Γ_0 (still denoted by P_i) the curve $\Gamma_0(\mathbb{C})$ has a description $z^2 - w^2$ with $P_1 = (0, 0)$ and with $f_0(z, w) = z$. Let U be a small neighborhood of $(0, 0)$ in \mathbb{C}^2 and $V \subset U$ a much smaller one. Let $U_0 = \Gamma_0(\mathbb{C}) \cap U$ and $V_0 = \Gamma_0(\mathbb{C}) \cap V$. For $P \neq P_i$ on U_0 we can use z to define a holomorphic coordinate at P . Consider a local deformation $z^2 - w^2 = t$ with $t \in \mathbb{C}$ and $|t|$ small and let U_t (resp. V_t) be the intersection with U (resp. V). We use a gluing of U_t and $\Gamma_0(\mathbb{C}) \setminus V_0$ as follows. For $z_0 \in \mathbb{C}^*$ let $z_0 \sqrt{z}$ be the locally defined holomorphic square root function such that $z_0 \sqrt{z_0^2} = z_0$. A point $Q \in U_0 \setminus V_0$ has coordinates (z, z) or $(z, -z)$. We identify (z, z) with $(z, z \sqrt{z^2 - t})$ and $(z, -z)$ with $(z, -z \sqrt{z^2 - t})$. (One should adopt the description of U_0 and V_0 to this identification). Moreover the projection $(z, w) \rightarrow z$ on U_t and $f_0(\mathbb{C})$ restricted to $\Gamma_0(\mathbb{C} \setminus V_0)$ glue together to a holomorphic mapping to $\mathbb{P}^1(\mathbb{C})$. At $\overline{P_i}$ (the image of $\overline{P_i}$ on Γ_0) one has a complex conjugated description $\overline{z}^2 - \overline{w}^2 = 0$ with $\overline{P_i} = (0, 0)$ and $f_0(\overline{z}, \overline{w}) = \overline{z}$. Using similar neighborhoods denoted by \overline{U} and \overline{V} one can consider a similar local smoothing $\overline{z}^2 - \overline{w}^2 = t'$. In case $t' = \overline{t}$ there is a complex conjugation between U_t and $\overline{U}_{\overline{t}}$ and it is compatible with the complex

conjugation on $\Gamma_0 \setminus (V_0 \cup \overline{V_0})$. Parameterizing this deformation by $(x, y) \in \mathbb{C}^2$ with $x = t + t'$ and $y = (1/i)(t - t')$ we obtain the deformation has a global antiholomorphic involution if $(x, y) \in \mathbb{R}^2$. We write $\Gamma_{(x,y)}$ for the deformation defined by (x, y) , it has a real structure for $(x, y) \in \mathbb{R}^2$. Moreover for $(x, y) \in \mathbb{R}^2$ the morphism $f_0 : \Gamma_0 \rightarrow \mathbb{P}^1$ deforms to a morphism $f_{(x,y)} : \Gamma_{(x,y)} \rightarrow \mathbb{P}^1$. Doing the deformation for $2 \leq i \leq s$ we obtain the existence of $\epsilon > 0$ such that there is a deformation $\pi : \mathcal{X} \rightarrow T = \{z \in \mathbb{C} : |z| < \epsilon\}$ and a T -morphism $F : \mathcal{X} \rightarrow T \times \mathbb{P}^1$ such that $\pi^{-1}(0) = \Gamma_0$, $F|_{\Gamma_0} : \Gamma_0 \rightarrow \mathbb{P}^1$ is f_0 and for $t \in T(\mathbb{R})$ the curve $\pi^{-1}(t) = X_t$ is a real curve of topological type $(g, s, 0)$ and $F|_{X_t} : X_t \rightarrow \mathbb{P}^1$ is a separating morphism of degree k . As a matter of fact, for $P \in \mathbb{P}^1(\mathbb{R})$ the fiber $f_t^{-1}(P)$ has degree k' on the component $C_{1,t}$ of $X_t(\mathbb{R})$ degenerating to $X'(\mathbb{R})$ and it has degree 1 on each component $C_{j,t}$ of $X_t(\mathbb{R})$ degenerating to the component $(\mathbb{P}^1)_j(\mathbb{R})$ of $\Gamma_0(\mathbb{R})$. It follows $\text{sepgon}(X_t) \leq k$ and we are going to prove that $\text{sepgon}(X_t) = k$ for $t \in \mathbb{R} \setminus \{0\}$ if $|t|$ is sufficiently small.

Let X_0 be the stable model of Γ_0 and let $\pi : \mathcal{C} \rightarrow S$ be a suited family for X_0 . Let $s_0 \in S(\mathbb{R})$ with $\pi^{-1}(s_0) = X_0$. Let ω_π be the relative dualizing sheaf of π . The restriction of ω_π to $\pi^{-1}(s_0)$ is the dualizing sheaf ω_{X_0} , it is very ample because of Lemma 5. Hence shrinking S we can assume $\omega_{\pi^{-1}(s)}$ is very ample for all $s \in S(\mathbb{C})$. Let $\mathcal{E} = \pi_*(\omega_\pi)$ a locally free sheaf of rank g on S defined over \mathbb{R} . Shrinking S we can assume \mathcal{E} is free and taking a base on it defined over \mathbb{R} we obtain an S -morphism $\mathcal{C} \rightarrow S \times \mathbb{P}^{g-1}$ and the image is a real family of canonically embedded curves. We use this family and we keep writing $\pi : \mathcal{C} \rightarrow S$ (so now this is a family of canonically embedded curves). From the previous arguments it follows there exists a real curve $T \subset S(\mathbb{R})$ containing s_0 such that for $t \in T \setminus \{s_0\}$ the curve $\pi^{-1}(t) = X_t$ is a curve of topological type $(g, s, 0)$ and it has a separating morphism $f_t : X_t \rightarrow \mathbb{P}^1$ of degree k . To make notations easier we write 0 to denote s_0 in the following arguments. We write $C_{1,t}$ to denote the components of $X_t(\mathbb{R})$ deforming to the smooth component $C_{1,0}$ of $X_0(\mathbb{R})$ and $C_{i,t}$ to denote the component of $X_t(\mathbb{R})$ deforming to S_i for $2 \leq i \leq s$.

Assume for all $t \in T \setminus \{0\}$ there exists a separating morphism $f_t : X_t \rightarrow \mathbb{P}^1$ of some degree $l < k$. Shrinking T if necessary there exists some $l < k$ such that for each $t \in T \setminus \{0\}$ there exists a separating morphism $f_t : X_t \rightarrow \mathbb{P}^1$ of degree l . Let D_t be a fiber of f_t , then $L_t = \mathcal{O}(D_t)$ is an invertible sheaf of degree l satisfying $h^0(L_t) \geq 2$. Since $l \leq g-1$ it follows from the Riemann-Roch Theorem that $h^0(\omega_{X_t}(-D_t)) \geq 1$. Hence there is a hyperplane section H of X_t (an effective canonical divisor) containing D_t . Let $E_t = H - D_t$. For each effective divisor D linearly equivalent to D_t there is a hyperplane section of X_t equal to $D + E_t$. From $h^0(\mathcal{O}_{X_t}(D_t)) \geq 2$, it follows there is a one-dimensional family of hyperplanes in \mathbb{P}^{g-1} containing E_t . The intersection of such hyperplanes is a linear subspace Λ_t of dimension at most $g-3$ containing E_t . (As a matter of fact the Riemann-Roch Theorem implies $\dim(\Lambda_t) = g-1-h^0(\mathcal{O}(D_t))$.) Those linear subspaces define a family of linear subspaces over $T \setminus \{0\}$ and this family has a limit Λ_0 above 0 (such limit exists because the Grassmannian is complete). Since $\Lambda_t \cap X_t$ is a scheme of length $2g-2-l$ for $t \neq 0$, in the limit $\Lambda_0 \cap X_0$ contains a scheme of length $2g-2-l$ (here we use the fact that the relative Hilbert scheme

is proper over the base). Let $C_{1,0}$ be the smooth component of $X_0(\mathbb{R})$ and let $P \in C_{1,0}$. Let P_t be a family of points over T such that $P_t \in C_{1,t}$ for $t \in T$ with $P_0 = P$. For $t \neq 0$ the fiber D_t of f_t containing P_t also contains some point $P_{t,j}$ of $C_{j,t}$ for $2 \leq j \leq s$. Let H_t be the hyperplane containing Λ_t such that $H_t \cap X_t = D_t + E_t$ (this is an intersection as scheme, hence possibly with multiple points). The limit of H_t is a hyperplane H_0 and the limit of $D_t + E_t$ is equal to the scheme $H_0 \cap X_0$. This subscheme contains $P \in C_{1,0}$. Moreover the limit of E_t is contained in $\Lambda_0 \cap X_0$ and the limit of D_t contains the limit of the points $P_{t,j}$ for $2 \leq j \leq s$, hence it contains S_j . This implies there is a subscheme of $H_0 \cap X_0$ of length at least $2g - 2 - l + s - 1$ having support contained in the finite set $(\Lambda_0 \cap X_0) \cup \{S_2, \dots, S_s\}$. Consider the normalization $n : X' \rightarrow X_0$ then $n^{-1}((\Lambda_0 \cap X_0) \cup \{S_2, \dots, S_s\})$ is a finite subset Z of X' . There are finitely many effective divisors of degree at most $2g - 2$ having support in Z . The embedding $X_0 \subset \mathbb{P}^{g-1}$ gives rise to a morphism $m : X' \rightarrow \mathbb{P}^{g-1}$ and since the image has degree $2g - 2$ this morphism corresponds to a subvector space V of dimension g of the space of global sections of an invertible sheaf \mathcal{L} of degree $2g - 2$ on X' . For each $P \in X'(\mathbb{R})$ we obtain a section in that vectorspace defining a divisor containing P and having a subdivisor of degree at least $2g - 3 - l + s$ having support contained in Z . Since $X'(\mathbb{R})$ has infinitely many points there exists an effective divisor D_0 of degree at least $2g - 3 - l + s$ having support on Z satisfying the following property. There exist infinitely many points P on $X'(\mathbb{R})$ such that there exist a section s of \mathcal{L} whose associated divisor D contains $D_0 + P$. For those divisors $D - D_0$ is an effective divisor corresponding to a global section of the invertible sheaf $\mathcal{L}(-D_0)$ on X' of degree at most $1 + l - s < 1 + k - s = k'$. Since there exist infinitely many points P on $X'(\mathbb{R})$ belonging to the divisor of a global section of $\mathcal{L}(-D_0)$ it implies $h^0(\mathcal{L}(-D_0)) \geq 1$. This contradicts the assumptions on X' .

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